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KÖTHE TÖEPLITZ DUALS OF CERTAIN BICOMPLEX SEQUENCE SPACES

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ABSTRACT

In this paper, we have investigated certain bicomplex duals of the class of bicomplex sequences defined by Srivastava & Srivastava [S2] and of the subclasses defined by Nigam [N1].Two types of duals namely α - dual and β - dual, given by Köthe & Töeplitz [K2], have been defined for bicomplex sequence spaces. Relations between these duals have been studied. Other two types of duals have also been defined. The inclusions have been shown to be proper by means of counter examples.

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KEYWORDS: Bicomplex Duals, Bicomplex Sequences, α - Dual, β - Dual

1. INTRODUCTION

About C_2

Bicomplex Numbers were introduced by Corrado Segre **(**1860 –1924**)** in1892**.** In [S1], he defined an infinite set of algebras and gave the concept of multi complex numbers. For the sake of brevity, we confine ourselves to the bicomplex version of his theory. The space of bicomplex numbers is the first in an infinite sequence of multicomplex spaces. The set of bicomplex numbers is denoted by C_2 and is defined as follows:

$$C_2 = \{x_1 + i_1 x_2 + i_2 x_3 + i_1 i_2 x_4 : x_1, x_2, x_3, x_4 \in C_0\}$$

Or equivalently

$$C_2 = \{z_1 + i_2 z_2 : z_1, z_2 \in C_1\}$$

where $i_1^{\,2} = i_2^{\,2} = -1$, $i_1 i_2 = i_2 i_1$ and C_0, C_1 denote the space of real and complex numbers respectively.

The binary compositions of addition and scalar multiplication on C_2 are defined coordinate wise and the multiplication in C_2 is defined term by term. With these binary compositions, C_2 becomes a commutative algebra with identity. Algebraic structure of C_2 differs from that of C_1 in many respects [P1]. Few of them, which pertain to our work, are mentioned below:

1.1 Idempotent Elements

Besides 0 and 1, there are exactly two nontrivial idempotent elements in C_2 defined as $e_1 = (1 + i_1 i_2) / 2$, $e_2 = (1 - i_1 i_2) / 2$.

Note that $e_1 + e_2 = 1$ and $e_1.e_2 = e_2.e_1 = 0$.

A bicomplex number $\xi = z_1 + i_2 z_2$ has a unique idempotent representation, [S3] as

$\xi = {}^1\xi\, e_1 + {}^2\xi\, e_2$ where ${}^1\xi = z_1 - i_1 z_2$, ${}^2\xi = z_1 + i_1 z_2$.

1.2 Two Principal Ideals

The Principal Ideals in C_2 generated by e_1 and e_2 are denoted by I_1 and I_2 respectively; thus

$$I_1 = \{\xi e_1 : \xi \in C_2\},$$

$$I_2 = \{\xi e_2 : \xi \in C_2\}.$$

Since $\xi = {}^1\xi e_1 + {}^2\xi e_2$, where ${}^1\xi$ and ${}^2\xi$ are the idempotent components of ξ, therefore these ideals can also be represented as

$$I_1 = \{(z_1 - i_1 z_2)e_1 : z_1, z_2 \in C_1\} = \{z\, e_1 : z \in C_1\}$$

$$I_2 = \{(z_1 + i_1 z_2)e_2 : z_1, z_2 \in C_1\} = \{z\, e_2 : z \in C_1\}$$

Note that $I_1 \cap I_2 = \{0\}$ and $I_1 \cup I_2 = O_2$, the set of all singular elements of C_2.

1.3 Zero Divisors

As we have seen, $e_1 . e_2 = e_2 . e_1 = 0$. Thus zero divisors exist in C_2. In fact, two Bicomplex numbers are divisors of zero if and only if one of them is a complex multiple of e_1 and the other is a complex multiple of e_2. In other words, two Bicomplex numbers are divisors of zero if and only if one of them is a member of $I_1 \sim \{0\}$ and the other is a member of $I_2 \sim \{0\}$.

1.4 Norm of a Bicomplex Number

The norm in C_2 is defined as

$$\|\xi\| = \{\, |z_1|^2 + |z_2|^2 \,\}^{1/2} = \left[\frac{|{}^1\xi|^2 + |{}^2\xi|^2}{2}\right]^{1/2}$$

C_2 becomes a modified Banach algebra with respect to this norm in the sense that

$$\|\xi.\eta\| \le \sqrt{2}\, \|\xi\| . \|\eta\|$$

1.5 Holomorphic Functions

Let X be a domain in C_2. A function $f : X \to C_2$ is said to be holomorphic function if $\forall \alpha \in X, \exists$ a discus $D(\alpha; r_1, r_2)$ with $r_1 > 0$, $r_2 > 0$ and a bicomplex power series representation in D such that

$$f(\xi) = \sum_{k \geq 1} \eta_k (\xi - \alpha)^k \ \forall \xi \in D(\alpha; r_1, r_2).$$

H(X) denotes the set of all holomorphic functions on C_2.

1.6 Entire Functions

A function f of a bicomplex variable is said to be an entire function if it is holomorphic in the entire bicomplexspace C_2.

1.7 Entire Bicomplex Sequence

If $f(\xi) = \sum_{k \geq 1} \alpha_k (\xi - \eta)^k$ represents an entire function, the series $\sum \alpha_k$ is called entire bicomplex series and the sequence $\{\alpha_k\}$ is called entire bicomplex sequence.

2. THE CLASSES B, B', B'' OF ENTIRE BICOMPLEX SEQUENCES

Srivastava & Srivastava [S2] defined the class B as

$$B = \left\{ f : f = \{\xi_k\} = \{ {}^1\xi_k e_1 + {}^2\xi_k e_2 \} : \sup_{k \geq 1} k^k \left| {}^1\xi_k \right| < \infty, \sup_{k \geq 1} k^k \left| {}^2\xi_k \right| < \infty \right\}$$

Every element of class B is the sequence of coefficients of an entire function and is, therefore, an entire bicomplex sequence

Two subclasses of the class B have been defined in [N1, W1] as follows:

$$B' = \left\{ f : f = \{ {}^1\xi_k e_1 \} : \sup_{k \geq 1} k^k \ \left| {}^1\xi_k \right| < \infty \right\}$$

$$B'' = \left\{ f : f = \{ {}^2\xi_k e_2 \} : \sup_{k \geq 1} k^k \left| {}^2\xi_k \right| < \infty \right\}$$

The elements of B' and B'' are the sequences with members in A_1 and A_2, respectively where A_1 and A_2 are the auxiliary space.

Note first that B' is closed with respect to the binary compositions induced on B' as a subset of B, owing to the consistency of idempotent representation and the algebraic structure of bicomplex numbers.

Norm in B' is defined as follows:

$$\| f \| = \sup_{k \geq 1} \left| {}^1\xi_k \right| , \quad f = \left\{ {}^1\xi_k . e_1 \right\} \in B'.$$

B′ is a Gelfand subalgebra of B (cf. [N1, W1]).

B′ is an algebra ideal of B which is not a maximal ideal [W2].

3. DUALS OF SEQUENCE SPACES

There are two types of dual of a sequence space, namely Algebraic dual and Topological dual. The set of all linear functionals, on a linear space V, with domain as V and range as K is denoted by L(V, K) = V#and is called algebraic dual of V. If we consider the set of all continuous linear functional, then we get topological dual denoted by V *. From the point of view of the duality theory, the study of sequence spaces is much more profitable. Köthe & Töeplitz were the first to recogonize the problem that it is difficult to find the topological duals of sequence spaces equipped with linear topologies. To resolve it, they introduced a kind of dual, α – dual, in quite many familiar and useful sequence spaces. In the same paper [K2], they also introduced another kind of dual namely β – dual which together with the given sequence space forms a nice dual system. A still more general notion of a dual, γ – dual was later introduced by Garling [G1]. For symmetric sequence spaces there is another notion of a dual, called a δ – dual due to Garling [G2] and Ruckle [R1].

Let λ be a sequence space, and ω is the family of all sequences $\{x_n\}$ with $x_n \in$ K, n ≥ 1. Define α - , β - , γ- , and δ – dual, respectively as follows:

1. $\alpha - dual \quad \lambda^{\alpha} = \left\{ x : x \in \omega, \sum_{i \geq 1} \left| x_i y_i \right| < \infty, \forall y \in \lambda \right\}$
2. $\beta - dual \quad \lambda^{\beta} = \left\{ x : x \in \omega, \left| \sum_{i \geq 1} x_i y_i \right| < \infty, \forall y \in \lambda \right\}$
3. $\gamma - dual \quad \lambda^{\gamma} = \left\{ x : x \in \omega, \sup_n \left| \sum_{i=1}^{n} x_i y_i \right| < \infty, \forall y \in \lambda \right\}$
4. $\delta - dual \quad \lambda^{\delta} = \left\{ x : x \in \omega, \sum_{i \geq 1} \left| x_i y_{\rho(i)} \right| < \infty, \forall y \in \lambda \, and \, \rho \in \pi \right\}$

Here π is the set of all permutations of N.

$\lambda^{\alpha}, \lambda^{\beta}, \lambda^{\gamma}, and\ \lambda^{\delta}$ are sequence spaces, and $\phi \subset \lambda^{\delta} \subset \lambda^{\alpha} \subset \lambda^{\beta} \subset \lambda^{\gamma}$.

OUR CONTRIBUTION

4. BICOMPLEX KÖTHE – TÖEPLITZ DUALS

If ω’ is the family of all bicomplex sequences $\xi = (\xi_k)$ with $\xi_k \in C_2$, k ≥ 1, where C_2 is the space of all bicomplex numbers. If λ be a bicomplex sequence space, then we denote α - , β - , γ- , and δ – duals of λ respectively by $\lambda_{\alpha}, \lambda_{\beta}, \lambda_{\gamma}\ and\ \lambda_{\delta}$.

These bicomplex duals are given as follows:

Definition 4.1 $\underline{\alpha - dual}$: $\lambda_\alpha = \left\{ \xi : \xi \in \omega', \sum_{i \geq 1} \left\| \xi_i \, \eta_i \right\| < \infty, \forall \eta_i \in \lambda \right\}$

Definition 4.2 $\underline{\beta - dual}$: $\lambda_\beta = \left\{ \xi : \xi \in \omega', \left\| \sum_{i \geq 1} \xi_i \, \eta_i \right\| < \infty, \forall \eta_i \in \lambda \right\}$

Definition 4.3 $\underline{\gamma - dual}$: $\lambda_\gamma = \left\{ \xi : \xi \in \omega', \sup_n \left\| \sum_{i=1}^{n} \xi_i \, \eta_i \right\| < \infty, \forall \eta_i \in \lambda \right\}$

Definition 4.4 $\underline{\delta - dual}$: $\lambda_\delta = \left\{ \xi : \xi \in \omega', \sum_{i \geq 1} \left\| \xi_i \, \eta_{\rho(i)} \right\| < \infty, \forall \eta_i \in \lambda \, and \, \rho \in \pi \right\}$

Where π is the set of all permutations of N.

Now we shall define α - dual of the class B and its subclasses B' and B''.

Definition 4.5: α – dual of the class B

$$B_\alpha = \left\{ (\eta_k) \in \omega' : \sum_{k \geq 1} \left\| \xi_k \, \eta_k \right\| < \infty, \forall (\xi_k) \in B \right\}$$

We know that a bicomplex series can be written in the form of its idempotent components, that is, as the sum of its idempotent component series which are series in complex space.

Symbolically we can write it as

$$\sum \xi_k \, \eta_k = \left[\sum {}^1\xi_k \, {}^1\eta_k \right] e_1 + \left[\sum {}^2\xi_k \, {}^2\eta_k \right] e_2$$

The series on the left hand side is convergent in C_2 if and only if both the series on the right hand side are convergent in C_1.

Thus α - dual can also be defined as follows:

$$B_\alpha = \left\{ (\eta_k) \in \omega' : \sum_{k \geq 1} \left| {}^1\xi_k \, {}^1\eta_k \right| < \infty, \sum_{k \geq 1} \left| {}^2\xi_k \, {}^2\eta_k \right| < \infty, \forall (\xi_k) \in B \right\}$$

Definition 4.6: α – dual of the class B'

$$(B')_\alpha = \left\{ (\eta_k) \in \omega' : \sum_{k \geq 1} \left\| \eta_k \, {}^1\xi_k \, e_1 \right\| < \infty, \forall \left({}^1\xi_k \, e_1 \right) \in B' \right\}$$

$$= \left\{ (\eta_k) \in \omega' : \sum_{k \geq 1} \left| {}^1\eta_k \, {}^1\xi_k \right| < \infty, \forall \left({}^1\xi_k \, e_1 \right) \in B' \right\}$$

Since, $\eta_k \, {}^1\xi_k \, e_1 = \left({}^1\eta_k \, e_1 + {}^2\eta_k \, e_2 \right) \left({}^1\xi_k \, e_1 + 0 \, e_2 \right) = {}^1\eta_k \, {}^1\xi_k \, e_1 + 0 \, e_2$

Note: we are not bothered about the second idempotent sequence i.e., $\left({}^{2}\xi_k\, e_2\right)$ of the sequence $\left(\xi_k\right)$.

Definition 4.7: α – dual of the class B''

$$(B'')_\alpha = \left\{ (\eta_k) \in \omega' : \sum_{k\ge 1} \left\| \eta_k\ {}^{2}\xi_k\, e_2 \right\| < \infty, \forall \left({}^{2}\xi_k\, e_2\right) \in B'' \right\}$$

$$= \left\{ (\eta_k) \in \omega' : \sum_{k\ge 1} \left| {}^{2}\eta_k\ {}^{2}\xi_k \right| < \infty, \forall \left({}^{2}\xi_k\, e_2\right) \in B'' \right\}$$

Since, $\eta_k\ {}^{2}\xi_k\, e_2 = \left({}^{1}\eta_k\, e_1 + {}^{2}\eta_k\, e_2\right)\left(0\, e_1 + {}^{2}\xi_k\, e_2\right) = 0\, e_1 + {}^{2}\eta_k\ {}^{2}\xi_k e_2$

Note: we are not bothered about the first idempotent sequence i.e., $\left({}^{1}\xi_k\, e_1\right)$ of the sequence $\left(\xi_k\right)$.

Theorem 4.1

A sequence (η_k) belongs to α - dual of the class B if and only if its first idempotent sequence belongs to α - dual of the class B’ and the second idempotent sequence belongs to α - dual of the class B’’, that is

$$(\eta_k) \in B_\alpha \Leftrightarrow \left({}^{1}\eta_k\, e_1\right) \in (B')_\alpha \ and\ \left({}^{2}\eta_k\, e_2\right) \in (B'')_\alpha .$$

Proof: Let $(\eta_k) \in B_\alpha$ be any sequence.

From def. 4.5,

$$(\eta_k) \in B_\alpha \Leftrightarrow \sum_{k\ge 1} \left\| \xi_k\, \eta_k \right\| < \infty, \forall (\xi_k) \in B$$

$$\Leftrightarrow \sum_{k\ge 1} \left| {}^{1}\xi_k\ {}^{1}\eta_k \right| .e_1 + \sum_{k\ge 1} \left| {}^{2}\xi_k\ {}^{2}\eta_k \right| .e_2 < \infty, \forall (\xi_k) \in B$$

$$\Leftrightarrow \sum_{k\ge 1} \left| {}^{1}\xi_k\ {}^{1}\eta_k \right| < \infty\ and\ \sum_{k\ge 1} \left| {}^{2}\xi_k\ {}^{2}\eta_k \right| < \infty, \forall (\xi_k) \in B$$

As we know, $(\xi_k) \in B \Rightarrow \left({}^{1}\xi_k\, e_1\right) \in B'\ and\ \left({}^{2}\xi_k\, e_2\right) \in B''$

Hence, $(\eta_k) \in B_\alpha \Leftrightarrow \left({}^{1}\eta_k\, e_1\right) \in (B')_\alpha\ and\ \left({}^{2}\eta_k\, e_2\right) \in (B'')_\alpha$, by def. 4.5, 4.6.

Now we shall study the inclusion relation between these duals. We shall prove that B_α is properly contained in $(B')_\alpha$.

Theorem 4.2:

(i) $B_\alpha \subset (B')_\alpha$.

(ii) $B_\alpha \subset (B'')_\alpha$

Proof: (i) We know that B' is a subset of B. By the known results, we also know that the dual of a set is contained in the dual of its subset, hence $B_\alpha \subseteq (B')_\alpha$

To prove the properness of this inclusion, we have to search for a sequence which belongs to $(B')_\alpha$ but does not belong to B_α.

Consider the sequence $(\eta_k) = \left(\frac{1}{k^2} e_1 + k^{2+k} e_2 \right)$.

First we show that $(\eta_k) \in (B')_\alpha$.

Let $\left({}^1\xi_k \, e_1 \right) \in B'$

$\Rightarrow \sup_k k^k \left| {}^1\xi_k \right| < \infty$

$\Rightarrow k^k \left| {}^1\xi_k \right| < M \,, \forall\, k \geq 1$ for some M.

$$\Rightarrow \left| {}^1\xi_k \right| < \frac{M}{k^k} \,, \forall\, k \geq 1 \tag{4.1}$$

$k^k > k^2 \,, \forall\, k \geq 2 \Rightarrow \frac{1}{k^k} < \frac{1}{k^2} \,, \forall\, k \geq 2$

And $\sum \frac{1}{k^2}$ is convergent $\Rightarrow \sum \frac{1}{k^k}$ is convergent, by comparison test.

$$\sum \frac{M}{k^k} \text{ is a convergent series.} \tag{4.2}$$

$\therefore$ From (4.1) and (4.2) $\sum \left| {}^1\xi_k \right|$ is convergent.

As in $(\eta_k) = \left(\frac{1}{k^2} e_1 + k^{2+k} e_2 \right)$, $\sum \left| {}^1\eta_k \right| = \sum \frac{1}{k^2}$ is convergent and $\sum \left| {}^1\xi_k \right|$ is also convergent, the series obtained by their term by term multiplication is always convergent.

$\therefore \sum \left| \frac{1}{k^2} \, {}^1\xi_k \right|$ is convergent, $\forall \left({}^1\xi_k \, e_1 \right) \in B'$.

$$\text{Thus } (\eta_k) \in (B')_\alpha . \tag{4.3}$$

To show that $(\eta_k) \notin B_\alpha$, we have to show that for some element $(\xi_k) \in B$, $\sum \left\| \xi_k \, \eta_k \right\|$ is not convergent.

Consider the sequence $(\xi_k) = k^{-k}\left(\frac{1}{k^2}e_1 + \frac{1}{k^3}e_2\right) = \left(k^{-k-2}e_1 + k^{-k-3}e_2\right)$.

Note first that,

$$\sup_k k^k \|\xi_k\| = \sup_k k^k \left(\frac{\left|{}^1\xi_k\right|^2 + \left|{}^2\xi_k\right|^2}{2}\right)^{1/2}$$

$$= \sup_k k^k \left(\frac{\left|k^{-2k-4}\right| + \left|k^{-2k-6}\right|}{2}\right)^{1/2}$$

$$= \sup_k \left\{\frac{1}{2}\left(\frac{1}{k^4} + \frac{1}{k^6}\right)\right\}^{1/2} < \infty$$

So that $(\xi_k) \in B$.

Now, $\sum_{k\geq 1} \|\xi_k\, \eta_k\| = \sum_{k\geq 1} \left\|\left(\frac{1}{k^{2+k}}e_1 + \frac{1}{k^{3+k}}e_2\right).\left(\frac{1}{k^2}e_1 + k^{2+k}e_2\right)\right\|$

$$= \sum_{k\geq 1} \left\|\left(\frac{1}{k^{4+k}}e_1 + \frac{1}{k}e_2\right)\right\|$$

$k^{4+k} > k^4 \Rightarrow \frac{1}{k^{4+k}} < \frac{1}{k^4}$ and $\sum_{k\geq 1}\left|\frac{1}{k^4}\right|$ is convergent therefore by comparison test $\sum_{k\geq 1}\left|\frac{1}{k^{4+k}}\right|$ is also convergent, but $\sum_{k\geq 1}\left|\frac{1}{k}\right|$ is not convergent.

Therefore, $(\eta_k) \notin B_\alpha$ (4.4)

Hence from (4.3) and (4.4) we get

$(\eta_k) \in (B')_\alpha$ But $(\eta_k) \notin B_\alpha$.

(ii) We know that B’ is a also a subset of B**.** We also know that $B_\alpha \subseteq (B'')_\alpha$

We can prove this part similarly as done in part (i) by taking

$$(\xi_k) = k^{-k}\left(\frac{1}{k^2}e_1 + \frac{1}{k^3}e_2\right) = \left(k^{-k-2}e_1 + k^{-k-3}e_2\right)$$

and $(\eta_k) = \left(k^{k+1}e_1 + \frac{1}{k^2}e_2\right)$.

Similar treatment can be given to other type of duals – namely β, γ and δ duals – defined by Kothe and Toeplitz [K2]. We define duals for the sake of ready reference.

Definition 4.8: β – dual of the class B

$$B_\beta = \left\{ (\eta_k) \in \omega' : \left\| \sum_{k\geq 1} \eta_k \ \xi_k \right\| < \infty, \forall (\xi_k) \in B \right\}$$

Or

$$\left\{ (\eta_k) \in \omega' : \left| \sum_{k\geq 1} {}^1\eta_k \ {}^1\xi_k \right| < \infty, \left| \sum_{k\geq 1} {}^2\eta_k \ {}^2\xi_k \right| < \infty, \forall (\xi_k) \in B \right\}$$

Definition 4.9: β – dual of the class B'

$$(B')_\gamma = \left\{ (\eta_k) \in \omega' : \left| \sum_{k\geq 1} {}^1\eta_k \ {}^1\xi_k \right| < \infty, \forall (\xi_k) \in B \right\}$$

Definition 4.10: β – dual of the class B''

$$(B')_\gamma = \left\{ (\eta_k) \in \omega' : \left| \sum_{k\geq 1} {}^2\eta_k \ {}^2\xi_k \right| < \infty, \forall (\xi_k) \in B \right\}$$

Theorem 4.3: A sequence (η_k) belongs to β - dual of the class B if and only if its first idempotent sequence belongs to β - dual of the class B' and the second idempotent sequence belongs to β - dual of the class B'', that is

$$(\eta_k) \in B_\beta \Leftrightarrow ({}^1\eta_k \ e_1) \in (B')_\beta \ and \ ({}^2\eta_k \ e_2) \in (B'')_\beta .$$

Proof: It can be proved easily in a similar way as we did in the proof of theorem 4.1.

Definition 4.11: γ – dual of the class B

$$B_\gamma = \left\{ (\eta_k) \in \omega' : \sup_n \left\| \sum_{k=1}^{n} \eta_k \ \xi_k \right\| < \infty, \forall (\xi_k) \in B \right\}$$

Or

$$\left\{ (\eta_k) \in \omega' : \sup_n \left| \sum_{k=1}^{n} {}^1\eta_k \ {}^1\xi_k \right| < \infty, \sup_n \left| \sum_{k=1}^{n} {}^2\eta_k \ {}^2\xi_k \right| < \infty, \forall (\xi_k) \in B \right\}.$$

Definition 4.12: γ – dual of the class B'

$$(B')_\gamma = \left\{ (\eta_k) \in \omega' : \sup_n \left| \sum_{k=1}^{n} {}^1\eta_k \, {}^1\xi_k \right| < \infty, \forall (\xi_k) \in B \right\}$$

Definition 4.13: γ – dual of the class B’’

$$(B'')_\gamma = \left\{ (\eta_k) \in \omega' : \sup_n \left| \sum_{k=1}^{n} {}^2\eta_k \, {}^2\xi_k \right| < \infty, \forall (\xi_k) \in B \right\}$$

Theorem 4.4: A sequence (η_k) belongs to γ - dual of the class B if and only if its first idempotent sequence belongs to γ - dual of the class B’ and the second idempotent sequence belongs to γ - dual of the class B’’, that is

$$(\eta_k) \in B_\gamma \Leftrightarrow ({}^1\eta_k \, e_1) \in (B')_\gamma \; and \; ({}^2\eta_k \, e_2) \in (B'')_\gamma .$$

Proof: It can be proved easily in a similar way as we did in the proof of theorem 4.1.

Definition 4.14: δ – dual of the class B"

$$= \left\{ (\eta_k) \in \omega' : \sum_{k \ge 1} \left| {}^1\eta_k \, {}^1\xi_{\rho(k)} \right| < \infty, \sum_{k \ge 1} \left| {}^2\eta_k \, {}^2\xi_{\rho(k)} \right| < \infty, \forall (\xi_k) \in B \, and \, \rho \in \pi \right\}$$

$$B_\delta = \left\{ (\eta_k) \in \omega' : \sum_{k \ge 1} \left\| \eta_k \, \xi_{\rho(k)} \right\| < \infty, \forall (\xi_k) \in B \, and \, \rho \in \pi \right\}$$

Definition 4.15: δ – dual of the class B'

$$(B')_\delta = \left\{ (\eta_k) \in \omega' : \sum_{k \ge 1} \left| {}^1\eta_k \, {}^1\xi_{\rho(k)} \right| < \infty, \forall (\xi_k) \in B \, and \, \rho \in \pi \right\}$$

Definition 4.16: δ – dual of the class B''

$$(B'')_\delta = \left\{ (\eta_k) \in \omega' : \sum_{k \ge 1} \left| {}^2\eta_k \, {}^2\xi_{\rho(k)} \right| < \infty, \forall (\xi_k) \in B \, and \, \rho \in \pi \right\}$$

Theorem 4.5: A sequence (η_k) belongs to δ - dual of the class B if and only if its first idempotent sequence belongs to δ - dual of the class B’ and the second idempotent sequence belongs to δ - dual of the class B’’, that is

$$(\eta_k) \in B_\delta \Leftrightarrow ({}^1\eta_k \, e_1) \in (B')_\delta \; and \; ({}^2\eta_k \, e_2) \in (B'')_\delta$$

Proof: It can be proved easily in a similar way as we did in the proof of theorem 4.1.

CONCLUSIONS

- Bicomplex Köthe – Toeplitz duals have been defined.
- There exist if and only if relation between α - dual of the class B and α - dual of its subclasses B’ and B’’ in terms of idempotent components. Similar relation holds for other three types of duals also.

- We also found that α - dual of the class B is properly contained in the α - dual of the subclass B' and α - dual of the subclass B''. And in both the cases, it is shown with the help of counter examples.

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REFERENCES

1. Garling, D.G.H.: "The β - and γ - duality", Proc. Cambridge Philos. Soc. (1967), 963 – 981.
2. Garling, D.G.H: "On symmetric sequence spaces", Proc. London Math. Soc. 16 (3), 1966, 85 – 106.
3. Kamthan, P.K & Gupta, M: "Sequence spaces and Series", Marcel Dekker, Inc. New York, 1981.
4. Köthe, G & Toeplitz, O.: "Lineare Raume mit unendlich vielen Koordinaten und Ringe unendlicher Matrizen", Jour. Reine angew. Math. 171, 1934, 193 – 226.
5. Larsen, R.: "Functional Analysis an Introduction" Marcel Dekker, New York, 1973.
6. Nigam, Mamta: "A study of certain sequence spaces of bicomplex numbers". Phil dissertation, Dr. B. R. Ambedkar University, Agra, 2008
7. Price, G. Baley: "An Introduction to Multicomplex spaces and" Marcel Dekker, Inc., 1991.
8. Ruckle, W: "On the characterization of sequence spaces associated with schauder bases" Studia Math., 28, 1967, 279 – 288
9. Segre C.: "Le Rappresentazioni Reali Delle forme complesse e Gli Enti Iperalgebrici" Math. Ann., 40, 1892, 413-467.
10. Srivastava, Rajiv K.: "On a class of Entire Bicomplex sequences" & Srivastava, Naveen K South East. Asian J. Math & Math. Sc. 5(3), 2007, 47-68.
11. Srivastava, Rajiv K.: "Certain topological aspects of Bicomplex space", Bull. Pure & Appl. Math. Dec., 2008.
12. Wagh, Mamta A: "On certain spaces of Bicomplex sequences" Inter. J. Phy. Chem. and Math. fund. 7(1), 2014.
13. Wagh, Mamta A.: "On a class of Bicomplex sequences" Inter. J. Trends in Math and Stats. 3(5), 2014.